Schwarzschild Space-Time in Gauge Theories of Gravity

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Abstract

In Poincaré gauge theory of gravity and in $\overline{\text{Poincaré}}$ gauge theory of gravity, we give the necessary and sufficient condition in order that the Schwarzschild space-time expressed in terms of the Schwarzschild coordinates is obtainable as a torsionless exact solution of gravitational field equations with a spinless point-like source having the energy-momentum density $\widetilde{\boldsymbol{T}}_{\mu}{}^{\nu}(x) = -Mc^2\delta_{\mu}{}^{0}\delta_{0}{}^{\nu}\delta^{(3)}(\boldsymbol{x})$. Further, for the case when this condition is satisfied, the energy-momentum and the angular momentum of the Schwarzschild space-time are examined in their relations to the asymptotic forms of vierbein fields. We show, among other things, that asymptotic forms of vierbeins are restricted by requiring the equality of the active gravitational mass and the inertial mass. Conversely speaking, this equality is violated for a class of vierbeins giving the Schwarzschild metric.

1 Introduction

There have been many attempts to describe gravity by gauge theory, among which we have Poincaré [1] and $\overline{\text{Poincaré}}$ [2] gauge theories. The former, which shall be written as

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PGT in short, is formulated by the requirement of invariance of the theory under the local Poincaré transformations. In this theory, the Poincaré group does not work as a gauge group in the sense of Yang-Mills theories. $\overline{\text{Poincaré}}$ gauge theory ($\bar{\text{P}}$GT) is formulated on the basis of the principal fiber bundle over the space-time possessing the covering group $\bar{P}_0$ of the Poincaré group as the structure group, by following the lines of the standard geometric formulation of Yang-Mills theories as closely as possible, in which the group $\bar{P}_0$ works as the gauge group in the sense of Yang-Mills theories.

The energy-momentum and angular momentum have been discussed in Ref. [3] for PGT and in Refs. [4] and [5] for $\bar{\text{P}}$GT, in the general framework of the formulations, by assuming that the vierbein fields and the Lorentz gauge fields approach their asymptotic values sufficiently rapidly.

In this paper, we examine these two theories. We study, among other things, the following: (1) We obtain the necessary and sufficient condition imposed on parameters in gravitational Lagrangian density in order that the Schwarzschild space-time expressed in terms of the Schwarzschild coordinates is obtainable as a torsionless exact solution of the gravitational field equations with a spinless point-like source located at the origin. (2) For the case for which this condition is satisfied, the energy-momentum and the angular momentum of the Schwarzschild space-time are examined in their relations to asymptotic behaviors of vierbein fields, and conditions for vierbeins to give the equality of the active gravitational mass and the inertial mass are obtained.

2 Preliminary

We briefly summarize the basics of Poincaré gauge theory and of $\overline{\text{Poincaré}}$ gauge theory for the convenience of latter discussion.

2.1 Poincaré gauge theory

In this theory, [1] the space-time is assumed to be a differentiable manifold endowed with the Lorentzian metric[1] $g_{\mu\nu}dx^\mu \otimes dx^\nu$. Here, $\{x^\mu; \mu = 0, 1, 2, 3\}$ is a local coordinate of

[1]In our conventions, the middle part of the Greek alphabet, $\mu, \nu, \lambda, \ldots$, refers to 0,1,2 and 3, while the initial part, $\alpha, \beta, \gamma, \ldots$, denotes 1,2 and 3. In a similar way, the middle part of the Latin alphabet, $i, j, k, \ldots$, means 0,1,2 and 3, unless otherwise stated. While the initial part, $a, b, c, \ldots$, denotes 1,2 and 3.

the space-time. Fundamental gravitational field variables are vierbeins $\boldsymbol{e}_k = e^{\mu}{}_k \partial/\partial x^{\mu}$ and Lorentz gauge potentials $A^{kl}{}_{\mu}$. For the duals $\boldsymbol{e}^k = e^k{}_{\mu} dx^{\mu}$ of the vierbeins, we have $g_{\mu\nu} = e^k{}_{\mu}\eta_{kl}e^l{}_{\nu}$ with $(\eta_{kl}) \stackrel{\rm def}{=} \mathrm{diag}(-1,1,1,1)$. The covariant derivative D_k of the field φ belonging to a representation σ of the Lorentz group is given by

$$D_k\varphi = e^{\mu}{}_k \left(\partial_{\mu}\varphi + \frac{i}{2}A^{lm}{}_{\mu}M_{lm}\varphi \right) , \tag{2·1}$$

where $M_{lm} \stackrel{\rm def}{=} -i\sigma_*(\overline{M}_{lm})$. Here, $\{\overline{M}_{lm}, l, m = 0,1,2,3\}$ is a basis of the Lie algebra of the Lorentz group satisfying the relation

$$[\overline{M}_{kl}, \overline{M}_{mn}] = -\eta_{km}\overline{M}_{ln} - \eta_{ln}\overline{M}_{km} + \eta_{kn}\overline{M}_{lm} + \eta_{lm}\overline{M}_{kn} , \tag{2·2}$$

$$\overline{M}_{kl} = -\overline{M}_{lk} , \tag{2·3}$$

and σ_* stands for the differential of σ.

The field strengths of $e^k{}_{\mu}$ and of $A^{kl}{}_{\mu}$ are given by

$$T^k{}_{lm} \stackrel{\rm def}{=} e^{\mu}{}_l e^{\nu}{}_m(\partial_{\mu}e^k{}_{\nu} - \partial_{\nu}e^k{}_{\mu}) + e^{\mu}{}_l A^k{}_{m\mu} - e^{\mu}{}_m A^k{}_{l\mu} , \tag{2·4}$$

$$R^{kl}{}_{mn} \stackrel{\rm def}{=} e^{\mu}{}_m e^{\nu}{}_n(\partial_{\mu}A^{kl}{}_{\nu} - \partial_{\nu}A^{kl}{}_{\mu} + A^k{}_{j\mu}A^{jl}{}_{\nu} - A^k{}_{j\nu}A^{jl}{}_{\mu}) , \tag{2·5}$$

respectively. The gauge potentials $A^{kl}{}_{\mu}$ and the affine connection coefficients $\Gamma^{\nu}_{\lambda\mu}$ are related through the relation

$$A^k{}_{l\mu} \equiv \Gamma^{\nu}_{\lambda\mu}e^k{}_{\nu}e^{\lambda}{}_l + e^k{}_{\nu}\partial_{\mu}e^{\nu}_l , \tag{2·6}$$

and we have

$$T^k{}_{\mu\nu} \equiv e^k{}_{\lambda}T^{\lambda}{}_{\mu\nu} , \quad R^k{}_{l\mu\nu} \equiv e^k{}_{\lambda}e^{\rho}{}_l R^{\lambda}{}_{\rho\mu\nu} \tag{2·7}$$

with

$$T^{\mu}{}_{\nu\lambda} \stackrel{\rm def}{=} \Gamma^{\mu}_{\lambda\nu} - \Gamma^{\mu}_{\nu\lambda} , \quad R^{\mu}{}_{\nu\lambda\rho} \stackrel{\rm def}{=} \partial_{\lambda}\Gamma^{\mu}_{\nu\rho} - \partial_{\rho}\Gamma^{\mu}_{\nu\lambda} + \Gamma^{\mu}_{\tau\lambda}\Gamma^{\tau}_{\nu\rho} - \Gamma^{\mu}_{\tau\rho}\Gamma^{\tau}_{\nu\lambda} . \tag{2·8}$$

The components $T^{\mu}{}_{\nu\lambda}$ and $R^{\mu}{}_{\nu\lambda\rho}$ are those of the torsion tensor and of the curvature tensor, respectively, and they are both non-vanishing in general.

The field components $e^k{}_{\mu}$ and $e^{\mu}{}_k$ will be used to convert Latin and Greek indices. Also, raising and lowering the Latin indices are accomplished with the aid of $(\eta^{kl}) \stackrel{\rm def}{=} (\eta_{kl})^{-1}$ and (η_{kl}), respectively.

For the matter field φ, $L_M(\varphi, D_k\varphi)$ is a Lagrangian[2] invariant under local Lorentz transformations and general coordinate transformations, if $L_M(\varphi, \partial_k\varphi)$ is an invariant Lagrangian on the Minkowski space-time.

The gravitational Lagrangian, which is invariant under local Lorentz transformations including also inversions and under general coordinate transformations and at most quadratic in torsion and curvature tensors, is given by

$$\bar{L}_G \stackrel{\text{def}}{=} L_T + L_R + aR \tag{2·9}$$

with

$$L_T \stackrel{\text{def}}{=} \alpha t^{klm} t_{klm} + \beta v^k v_k + \gamma a^k a_k \ , \tag{2·10}$$

$$\begin{aligned} L_R \stackrel{\text{def}}{=}\ & c_1 A^{klmn} A_{klmn} + c_2 B^{klmn} B_{klmn} \\ & + c_3 C^{klmn} C_{klmn} + c_4 E^{kl} E_{kl} + c_5 I^{kl} I_{kl} + c_6 R^2 \ . \end{aligned} \tag{2·11}$$

In the above, $\alpha, \beta, \gamma, c_k$ $(k = 1, 2, \cdots, 6)$ and a are real constants. Also, t_{ijk}, v_k and a_k are irreducible components of the field strength T_{klm}, and $A_{klmn}, B_{klmn}, C_{klmn}, E_{kl}$, I_{kl} and R are irreducible components of the field strength R_{klmn}. These components are given explicitly in Appendix A. Then,

$$\boldsymbol{I} \stackrel{\text{def}}{=} \frac{1}{c} \int \bar{\boldsymbol{L}} d^4x \tag{2·12}$$

is the total action of the system, where c is the light velocity in the vacuum and $\bar{\boldsymbol{L}}$ is defined by

$$\bar{\boldsymbol{L}} \stackrel{\text{def}}{=} \sqrt{-g}[\bar{L}_G + L_M(\varphi, D_k\varphi)] \tag{2·13}$$

with $g \stackrel{\text{def}}{=} \det(g_{\mu\nu})$. The parameter a is fixed [6][3] to be $a = 1/2\kappa = c^4/16\pi G$, which we shall assume to hold in this paper, by the requirement that the theory has the Newtonian limit. Here, κ and G stand for the Einstein gravitational constant and the Newton gravitational constant, respectively.

When the torsion is vanishing, $T_{klm} \equiv 0$, the field equations $\delta\bar{\boldsymbol{L}}/\delta e^i{}_\mu = 0$ and $\delta\bar{\boldsymbol{L}}/\delta A^{ij}{}_\mu = 0$ reduce to[4]

$$2aG_{ij}(\{\}) + (3c_2 + 2c_5)\Big[R_{im}(\{\})R_j{}^m(\{\}) + R^{mn}(\{\})(R_{imjn}(\{\})$$

[2]From now on, Lagrangian density is simply called Lagrangian.

[3]The theory has the Newtonian limit, also when [6] $c_k = \infty$ $(k = 1, 2, \cdots, 6)$ and $\alpha + 4\beta + 9\alpha\beta\kappa = 0$. We do not deal with this extreme case in the present paper.

[4]$A_{\dots[k\dots l]\dots} \stackrel{\text{def}}{=} \frac{1}{2}(A_{\dots k \dots l \dots} - A_{\dots l \dots k \dots})$.

$$
-\frac{1}{2}\eta_{ij}R_{mn}(\{\}))\Big] - (2c_2 + c_5 - 4c_6)R(\{\})(R_{ij}(\{\}) - \frac{1}{4}\eta_{ij}R(\{\})) \quad = \quad T_{ij} \ , \tag{2·14}
$$

$$
(3c_2 + 2c_5)\nabla_{[i}G_{j]k}(\{\}) + (c_2 + c_5 + 4c_6)\eta_{k[i}\partial_{j]}G(\{\}) = -S_{ijk} \ , \tag{2·15}
$$

respectively. Here, T_{ij} and S_{ijk} are the energy-momentum and spin densities of the source field φ, respectively. They are defined by

$$
T_{ij} \overset{\text{def}}{=} \frac{1}{\sqrt{-g}} e_{j\mu} \frac{\delta \boldsymbol{L}_M}{\delta e^i{}_\mu} \ , \tag{2·16}
$$

$$
S_{ijk} \overset{\text{def}}{=} -\frac{1}{\sqrt{-g}} e_{k\mu} \frac{\delta \boldsymbol{L}_M}{\delta A^{ij}{}_\mu} \tag{2·17}
$$

with $\boldsymbol{L}_M \overset{\text{def}}{=} \sqrt{-g}L_M$. Also, $G_{ij}(\{\})$ and $G(\{\})$ stand for the Einstein tensor and its trace, respectively:

$$
G_{ij}(\{\}) \overset{\text{def}}{=} R_{ij}(\{\}) - \frac{1}{2}\eta_{ij}R(\{\}) \ , \quad G(\{\}) \overset{\text{def}}{=} \eta^{ij}G_{ij}(\{\}) \ , \tag{2·18}
$$

where we have defined

$$
R_{ij}(\{\}) \overset{\text{def}}{=} e^\mu{}_i e^\nu{}_j R^\lambda{}_{\mu\lambda\nu}(\{\}) \ , \quad R(\{\}) \overset{\text{def}}{=} \eta^{ij}R_{ij}(\{\}) \tag{2·19}
$$

with the Riemann-Christoffel curvature tensor

$$
R^\lambda{}_{\rho\mu\nu}(\{\}) \overset{\text{def}}{=} \partial_\mu \begin{Bmatrix} \lambda \\ \rho\ \nu \end{Bmatrix} - \partial_\nu \begin{Bmatrix} \lambda \\ \rho\ \mu \end{Bmatrix} + \begin{Bmatrix} \lambda \\ \sigma\ \mu \end{Bmatrix}\begin{Bmatrix} \sigma \\ \rho\ \nu \end{Bmatrix} - \begin{Bmatrix} \lambda \\ \sigma\ \nu \end{Bmatrix}\begin{Bmatrix} \sigma \\ \rho\ \mu \end{Bmatrix} \ . \tag{2·20}
$$

2.2 $\overline{\text{Poincaré}}$ gauge theory

This theory [2] is formulated on the basis of the principal fiber bundle over the space-time possessing the covering group $\bar{P}_0$ of the proper orthochronous Poincaré group as the structure group. The fundamental field variables are the translational gauge potentials $A^k{}_\mu$, the Lorentz gauge potentials $A^{kl}{}_\mu$, Higgs-type field $\psi = \{\psi^k\}$ and matter field φ. This theory is different from Poincaré gauge theory in various respects, among which the following is remarkable:

(A) There is the non-dynamical Higgs-type field $\psi = \{\psi^k\}$ playing key roles in the formulation. Its existence is a necessary consequence of a basic postulate on the space-time manifold and of the structure of the group $\bar{P}_0$. Also, ψ is directly related to the existence of a subbundle which gives a spinor structure on the space-time, and the spinor structure is built in as a sub-structure of the formulation. Its field equation is automatically satisfied, if those of $A^k{}_\mu$ and of φ are both satisfied.

(B) Dual components $e^k{}_\mu$ of vierbeins are related to the field ψ and the gauge potentials $A^k{}_\mu, A^{kl}{}_\mu$ through the relation

$$e^k{}_\mu = \partial_\mu \psi^k + A^k{}_{l\mu}\psi^l + A^k{}_\mu \ . \tag{2·21}$$

The components $e^k{}_\mu$ are invariant under *internal* translations.

(C) The field strength $R^k{}_{\mu\nu}$ of $A^k{}_\mu$ is given by

$$R^k{}_{\mu\nu} \stackrel{\text{def}}{=} \partial_\mu A^k{}_\nu - \partial_\nu A^k{}_\mu + A^k{}_{l\mu}A^l_\nu - A^k{}_{l\nu}A^l_\mu \ , \tag{2·22}$$

and we have the relation

$$T^k{}_{\mu\nu} = R^k{}_{\mu\nu} + R^k{}_{l\mu\nu}\psi^l \ . \tag{2·23}$$

The field strengths $T^k{}_{\mu\nu}$ and $R^{kl}{}_{\mu\nu}$ are both invariant under *internal* translations.

(D) A nonintegrable phase factor describes both of the motions of a point and of a vector. [7]

(E) Generators of internal Poincaré transformations and of affine coordinate transformations depend on the choice of the set of independent fields variables, i.e., generators for the case when $\{\psi^k, A^k{}_\mu, A^{kl}{}_\mu, \varphi\}$ is chosen as the set of independent field variables are different from the corresponding ones for the case when $\{\psi^k, e^k{}_\mu, A^{kl}{}_\mu, \varphi\}$ is chosen instead. When $\{\psi^k, A^k{}_\mu, A^{kl}{}_\mu, \varphi\}$ is employed as the set, we have the following: For suitable asymptotic forms of field variables at spatial infinity, *the conserved total energy-momentum and the total* (=*spin*+*orbital*) *angular momentum of an isolated system are obtained as the generators of the internal Poincaré transformations, and the generators of the general affine coordinate transformations vanish.* [4, 5]

(F) It admits the possible existence of matter fields having non-vanishing "intrinsic" energy-momentum P_k (=the quantum number associated with the internal translation), [8] and the covariant derivative of the matter field φ takes the form

$$D_k\varphi = e^\mu{}_k \left(\partial_\mu\varphi + \frac{i}{2}A^{lm}{}_\mu M_{lm}\varphi + iA^l{}_\mu P_l\varphi\right) \ , \tag{2·24}$$

in general.

(G) An extended new general relativity is obtained as a reduction of the theory. [9]

The invariance of the action integral under internal $\bar{P}_0$ gauge transformations requires the gravitational Lagrangian to be a function of T_{klm} and of R_{klmn}, and the gravitational Lagrangian agrees with that in PGT. Hence, *gravitational field equations take the same forms in these theories.*

3 Schwarzschild space-time

By Schwarzschild space-time, we mean a space-time having the vanishing torsion and the Schwarzschild metric which has the expression

$$ds^2 = -\left(1 - \frac{r_0}{r}\right)(dx^0)^2 + \left(1 - \frac{r_0}{r}\right)^{-1}(dr)^2 + r^2\left[(d\theta)^2 + \sin^2\theta(d\phi)^2\right] , \tag{3·1}$$

in the Schwarzschild coordinates (x^0, r, θ, ϕ). Here, r_0 is the Schwarzschild radius $r_0 = 2GM/c^2$ with M being the active gravitational mass of the central gravitating body. A static chargeless mass point with the active gravitational mass M is considered to produce the Schwarzschild space-time, and the energy-momentum density $\widetilde{\boldsymbol{T}}_\mu{}^\nu$ of the gravitational source is expected to have the expression

$$\widetilde{\boldsymbol{T}}_\mu{}^\nu(x) = -Mc^2\delta_\mu{}^0\delta_0{}^\nu\delta^{(3)}(\boldsymbol{x}) . \tag{3·2}$$

Both in PGT and in $\bar{\mathrm{P}}$GT, the Schwarzschild space-time is obtainable as an exact solution of the gravitational field equations with $S_{ijk} \equiv 0$ and $\widetilde{\boldsymbol{T}}_\mu{}^\nu(x) \stackrel{\mathrm{def}}{=} \sqrt{-g}e^k{}_\mu e^{\nu l}T_{kl} = -Mc^2\delta_\mu{}^0\delta_0{}^\nu\delta^{(3)}(\boldsymbol{x})$, if [5] and only if

$$3c_2 + 2c_5 = 0 = c_5 + 12c_6 , \tag{3·3}$$

as we shall show below. Since $T_{klm} \equiv 0$ for this space-time, the curvature R_{ijmn} agrees with the Riemann-Christoffel curvature $R_{ijmn}(\{\})$, and it is given by

$$\begin{aligned}
R_{(0)(a)(0)(b)} &= R_{(0)(a)(0)(b)}(\{\}) = -\frac{1}{2}\frac{h'}{r}\delta_{(a)(b)} + \left(-\frac{h''}{2} + \frac{1}{2}\frac{h'}{r}\right)\frac{x^{(a)}x^{(b)}}{r^2} , \\
R_{(0)(a)(b)(c)} &= R_{(0)(a)(b)(c)}(\{\}) = 0 ,
\end{aligned}$$

[5] In Ref. [1], the following has been shown: (1) For a gravitational source with $S_{ijk} \equiv 0$, *any* solution of the Einstein equation is also a solution of the gravitational field equations in PGT, if the condition (3·3) is satisfied. (2) The gravitational field equation $\delta\bar{\boldsymbol{L}}/\delta A^{ij}{}_\mu = 0$ contains third derivatives of the metric tensor, unless the condition (3·3) is satisfied.

$$
\begin{aligned}
R_{(a)(b)(c)(d)} &= R_{(a)(b)(c)(d)}(\{\}) = \left(\delta_{(a)(c)}\delta_{(b)(d)} - \delta_{(a)(d)}\delta_{(b)(c)}\right)\frac{h}{r^2} \\
&+ \left(\frac{x^{(a)}x^{(c)}}{r^2}\delta_{(b)(d)} - \frac{x^{(a)}x^{(d)}}{r^2}\delta_{(b)(c)}\right. \\
&\quad \left. + \frac{x^{(b)}x^{(d)}}{r^2}\delta_{(a)(c)} - \frac{x^{(b)}x^{(c)}}{r^2}\delta_{(a)(d)}\right)\left(\frac{1}{2}\frac{h'}{r} - \frac{h}{r^2}\right)
\end{aligned} \tag{3·4}
$$

with $h \stackrel{\text{def}}{=} r_0/r$, where we have written Lorentz (Latin) indices in parentheses. Substituting Eq. (3·4) into the field equation (2·14), we obtain:

$$
\begin{aligned}
\widetilde{\boldsymbol{T}}_0{}^0 &= -2a\left(\frac{h'}{r} + \frac{h}{r^2}\right) + (3c_2 + 2c_5)\left[\left(\frac{h''}{2} + \frac{h'}{r}\right)\frac{h''}{2} - \left(\frac{h'}{r} + \frac{h}{r^2}\right)\frac{h}{r^2}\right] \\
&+ (2c_2 + c_5 - 4c_6)\left[-\left(\frac{h''}{2} + \frac{h'}{r}\right)^2 + \left(\frac{h'}{r} + \frac{h}{r^2}\right)^2\right], \\
\widetilde{\boldsymbol{T}}_0{}^\alpha &= 0 = \widetilde{\boldsymbol{T}}_\alpha{}^0, \\
\widetilde{\boldsymbol{T}}_\alpha{}^\beta &= -2a\left[\delta_\alpha{}^\beta\left(\frac{h''}{2} + \frac{h'}{r}\right) - \frac{x^\alpha x^\beta}{r^2}\left(\frac{h''}{2} - \frac{h}{r^2}\right)\right] \\
&- (3c_2 + 2c_5)\left(\delta_\alpha{}^\beta - 2\frac{x^\alpha x^\beta}{r^2}\right)\left[\left(\frac{h''}{2} + \frac{h'}{r}\right)\frac{h''}{2} - \left(\frac{h'}{r} + \frac{h}{r^2}\right)\frac{h}{r^2}\right] \\
&+ (2c_2 + c_5 - 4c_6)\left(\delta_\alpha{}^\beta - 2\frac{x^\alpha x^\beta}{r^2}\right)\left[\left(\frac{h''}{2} + \frac{h'}{r}\right)^2 - \left(\frac{h'}{r} + \frac{h}{r^2}\right)^2\right].
\end{aligned} \tag{3·5}
$$

We regularize [10] the function $h = r_0/r$ as $r_0/\sqrt{r^2+\epsilon^2}$, in order to make a distribution theoretical treatment, then the regularized energy density $\widetilde{\boldsymbol{T}}_0{}^0(x;\epsilon)$ of the gravitational source takes the form

$$
\widetilde{\boldsymbol{T}}_0{}^0(x;\epsilon) = -\frac{2ar_0\epsilon^2}{r^2(r^2+\epsilon^2)^{3/2}} + \Lambda(r;\epsilon). \tag{3·6}
$$

Here, we have defined

$$
\begin{aligned}
\Lambda(r;\epsilon) &\stackrel{\text{def}}{=} (c_2 + c_5 + 4c_6)r_0^2\left[\frac{9}{4}\frac{\epsilon^4}{(r^2+\epsilon^2)^5} - \frac{\epsilon^4}{r^4(r^2+\epsilon^2)^3}\right] \\
&\quad - (3c_2 + 2c_5)r_0^2\left[\frac{3}{2}\frac{\epsilon^2}{(r^2+\epsilon^2)^4} + \frac{\epsilon^2}{r^2(r^2+\epsilon^2)^3}\right].
\end{aligned} \tag{3·7}
$$

The first term in the right hand side (r.h.s.) of Eq. (3·6) has the well-defined limit,

$$
\lim_{\varepsilon\to 0}\left\{-\frac{2ar_0\epsilon^2}{r^2(r^2+\epsilon^2)^{3/2}}\right\} = -Mc^2\delta^{(3)}(\boldsymbol{x}), \tag{3·8}
$$

where we have used $a = c^4/16\pi G$. For the second term $\Lambda(r;\varepsilon)$, we have

$$\int_0^\infty \Lambda(r;\epsilon) r^2 dr = -\frac{r_0^2}{\epsilon^3}(c_2 + c_5 + 4c_6)\int_0^\infty \frac{dx}{x^2} + \frac{15\pi}{1024}\frac{r_0^2}{\epsilon^3}(19c_2 + 35c_5 + 268c_6) \ , \qquad (3\cdot9)$$

in which the integral in the r.h.s. is diverging. As is known from Eqs. (3·5), (3·6), (3·8) and (3·9), we have the limit

$$\widetilde{\boldsymbol{T}}_\mu{}^\nu(x) = \lim_{\epsilon\to 0}\widetilde{\boldsymbol{T}}_\mu{}^\nu(x;\epsilon) = -Mc^2\delta_\mu{}^0\delta_0{}^\nu\delta^{(3)}(\boldsymbol{x}) \ , \qquad (3\cdot10)$$

if and only if the condition

$$c_2 + c_5 + 4c_6 = 0 = 19c_2 + 35c_5 + 268c_6 \ , \qquad (3\cdot11)$$

which is equivalent to Eq. (3·3), is satisfied.[6] Obviously, the field equation (2·15) with $S_{ijk} \equiv 0$ is satisfied identically, if the condition (3·3) is satisfied.[1]

From the above discussion and from the fact mentioned in the footnote on page 7, we know that the Lagrangian L_R with the condition (3·3), which we shall employ in the following, is favorable in various respects.

4 Spherically symmetric vierbeins giving the Schwarzschild metric

In view of the fact that the Schwarzschild space-time is asymptotically Minkowskian, [3] we choose vierbeins satisfying

$$\lim_{r\to\infty} e^k{}_\mu = e^{(0)k}{}_\mu \qquad (4\cdot1)$$

with $e^{(0)k}{}_\mu$ being constants satisfying $e^{(0)k}{}_\mu \eta_{kl} e^{(0)l}{}_\nu = \eta_{\mu\nu}$. The general forms of components $e^k{}_\mu$ having spherical symmetry have been given by Robertson, [11] which can be written as

$$\begin{aligned} e^{(0)}{}_0 &= A \ , \quad e^{(0)}{}_\alpha = B\frac{x^\alpha}{r} \ , \quad e^{(a)}{}_0 = C\frac{x^{(a)}}{r} \ , \\ e^{(a)}{}_\alpha &= D\delta^{(a)}{}_\alpha + E\frac{x^{(a)}x^\alpha}{r^2} + F\epsilon_{(a)\alpha\beta}\frac{x^\beta}{r} \ . \end{aligned} \qquad (4\cdot2)$$

[6] As for energy-momentum densities of the gravitational field, see §5. for PGT and §6. for $\bar{\text{P}}$GT, respectively.

Here, A, B, C, D, E and F are functions of $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$ and of x^0, and $\epsilon_{(a)(b)(c)}$ stands for the three-dimensional Levi-Civita symbol with $\epsilon_{(1)(2)(3)} = 1$. These give the Schwarzschild metric (3·1), if and only if

$$\begin{aligned} A^2 - C^2 &= 1 - h \ , \quad AB - C(D+E) = 0 \ , \quad D^2 + F^2 = 1 \ , \\ -B^2 + (D+E)^2 &= (1-h)^{-1} \ . \end{aligned} \tag{4·3}$$

The six functions $A, B, \cdots$ and F are required to satisfy the four relations in Eq. (4·3). Thus, if A and D are fixed, the others are uniquely determined up to the sign. Equations (4·1), (4·2) and (4·3) lead to

$$\begin{aligned} \lim_{r\to\infty} A &= A_\infty \ , \quad \lim_{r\to\infty} D = D_\infty \ , \\ \lim_{r\to\infty} B &= \lim_{r\to\infty} C = \lim_{r\to\infty} E = \lim_{r\to\infty} F = 0 \end{aligned} \tag{4·4}$$

with A_∞ and D_∞ being constants such that

$$(A_\infty)^2 = 1 = (D_\infty)^2 \ . \tag{4·5}$$

From Eq. (4·3), it follows that

$$B(1-h) = \Delta C \ , \ \ (D+E)(1-h) = \Delta A \ , \ \ \Delta^2 = 1 \tag{4·6}$$

with $\Delta \stackrel{\mathrm{def}}{=} A(D+E) - BC$.

In what follows, we shall choose, for simplicity, the gauge of internal Lorentz group in a way such that A and hence B and C are independent of [7] x^0.

In Appendix B, we give an example of the set of spherically symmetric vierbeins which gives the Schwarzschild metric and has a x^0-independent A.

5 Generators in Poincaré gauge theory

In this section, following Hayashi and Shirafuji, [3] we examine generators of Lorentz gauge transformations and of Poincaré coordinate transformations in the Schwarzschild space-time. We employ, instead of $\bar{\boldsymbol{L}}_G$, the following $\boldsymbol{L}_G$

$$\boldsymbol{L}_G \stackrel{\mathrm{def}}{=} \bar{\boldsymbol{L}}_G + 2a\partial_\nu(\sqrt{-g}\ e^\mu{}_k e^\nu{}_l A^{kl}{}_\mu) \ . \tag{5·1}$$

[7] This gauge choice is always possible for an arbitrarily given spherically symmetric vierbeins giving the Schwarzschild metric.

as the gravitational Lagrangian. Thus, the total Lagrangian becomes

$$\boldsymbol{L} \stackrel{\text{def}}{=} \boldsymbol{L}_G + \boldsymbol{L}_M \ . \tag{5·2}$$

The divergence term in the r.h.s. of Eq. (5·1) is to get reasonable generators, [3, 4] and it does not affect field equations.

The canonical energy-momentum M_μ, the orbital angular momentum $L^{\mu\nu}$ and the spin angular momentum S_{ij} are generators of coordinate translations, of Lorentz coordinate transformations and of *internal* Lorentz transformations, respectively, and they are expressed as, [3][8]

$$\begin{aligned}
M_\mu &\stackrel{\text{def}}{=} \int_\sigma {}^{\text{tot}}\widetilde{\boldsymbol{T}}_\mu{}^\nu d\sigma_\nu \ , && (5·3)\\
L^{\mu\nu} &\stackrel{\text{def}}{=} \eta^{\nu\rho} \int_\sigma \left(x^\mu \ {}^{\text{tot}}\widetilde{\boldsymbol{T}}_\rho{}^\lambda - \boldsymbol{\Psi}_\rho{}^{\mu\lambda} \right) d\sigma_\lambda - (\mu \rightleftarrows \nu) \ , && (5·4)\\
S_{ij} &\stackrel{\text{def}}{=} \int_\sigma {}^{\text{tot}}\widetilde{\boldsymbol{S}}_{ij}{}^\nu d\sigma_\nu && (5·5)
\end{aligned}$$

with

$$\begin{aligned}
{}^{\text{tot}}\widetilde{\boldsymbol{T}}_\mu{}^\nu &\stackrel{\text{def}}{=} \widetilde{\boldsymbol{T}}'_\mu{}^\nu + \widetilde{\boldsymbol{t}}_\mu{}^\nu \ , && (5·6)\\
\boldsymbol{\Psi}_\mu{}^{\nu\rho} &\stackrel{\text{def}}{=} \frac{\partial \boldsymbol{L}}{\partial e^k{}_{\nu,\rho}} e^k{}_\mu + \frac{\partial \boldsymbol{L}}{\partial A^{kl}{}_{\nu,\rho}} A^{kl}{}_\mu \ , && (5·7)\\
{}^{\text{tot}}\widetilde{\boldsymbol{S}}_{ij}{}^\nu &\stackrel{\text{def}}{=} -2 \frac{\partial \boldsymbol{L}}{\partial e^{[i}{}_{\lambda,\nu}} e_{j]\lambda} - 4 \frac{\partial \boldsymbol{L}}{\partial A^{[ik}{}_{\lambda,\nu}} A_{j]}{}^k{}_\lambda - i \frac{\partial \boldsymbol{L}}{\partial \varphi_{,\mu}} M_{ij} \varphi \ . && (5·8)
\end{aligned}$$

Here, σ and $d\sigma_\nu$ stand for a space-like surface and the surface element on it, respectively, and we have defined[9]

$$\widetilde{\boldsymbol{T}}'_\mu{}^\nu \stackrel{\text{def}}{=} \delta_\mu{}^\nu \boldsymbol{L}_M - \frac{\partial \boldsymbol{L}_M}{\partial \varphi_{,\nu}} \varphi_{,\mu} \ , \tag{5·9}$$

[8]In general, integrands in representations (5·3), (5·4), (5·5), etc. of physical quantities are singular at $\boldsymbol{x} = \boldsymbol{0}$ and on the horizon $r = r_0$. But, for the case of the example given in Appendix B, this cause no trouble, if $K(x^0) = \text{constant} > 0$, For the energy-momentum M_μ, for example, this can be confirmed easily by using Eqs. (5·3), (5·12), (5·20), (B·1) $\sim$ (B·5). In what follows, we restrict ourselves to cases such that the singularities at $\boldsymbol{x} = \boldsymbol{0}$ and on the horizon $r = r_0$ cause no trouble (see also the footnote 11 on page 17).

[9]The gravitational energy-momentum density $\widetilde{\boldsymbol{t}}_\mu{}^\nu$ is not vanishing, even if the condition (3·3) is satisfied. The integrated gravitational energy-momentum, however, vanishes, when the conditions (3·3) and (5·25) are both satisfied. This density does not reduce to the gravitational energy-momentum density employed in Ref. [10], even for the case with $c_k = 0$ $(k = 1, 2, \cdots, 6)$ and with $T_{klm} \equiv 0$ for which the relation $\bar{L}_G = aR(\{\})$ holds.

$$\widetilde{\boldsymbol{t}}_{\mu}{}^{\nu} \stackrel{\rm def}{=} \delta_{\mu}{}^{\nu}\boldsymbol{L}_G - \frac{\partial \boldsymbol{L}_G}{\partial e^k{}_{\lambda,\nu}} e^k{}_{\lambda,\mu} - \frac{\partial \boldsymbol{L}_G}{\partial A^{kl}{}_{\lambda,\nu}} A^{kl}{}_{\lambda,\mu} \ . \tag{5·10}$$

There is the relation,

$$\widetilde{\boldsymbol{T}}'_{\mu}{}^{\nu} = \widetilde{\boldsymbol{T}}_{\mu}{}^{\nu} \ , \tag{5·11}$$

which follows from the fact that $\boldsymbol{L}_M$ is a scalar density. Also, we have the identities

$$-\frac{\delta \boldsymbol{L}}{\delta e^k{}_\nu} e^k{}_\mu - \frac{\delta \boldsymbol{L}}{\delta A^{kl}{}_\nu} A^{kl}{}_\mu + {}^{\rm tot}\widetilde{\boldsymbol{T}}_{\mu}{}^{\nu} \equiv \partial_\lambda \boldsymbol{\Psi}_{\mu}{}^{\nu\lambda} \ , \tag{5·12}$$

$$\frac{\delta \boldsymbol{L}}{\delta A^{ij}{}_\mu} + \frac{1}{2} {}^{\rm tot}\widetilde{\boldsymbol{S}}_{ij}{}^{\mu} \equiv \frac{1}{2}\partial_\nu \boldsymbol{\Sigma}_{ij}{}^{\mu\nu} \ , \tag{5·13}$$

where we have defined

$$\boldsymbol{\Sigma}_{ij}{}^{\mu\nu} \stackrel{\rm def}{=} -2\left[\frac{\partial \boldsymbol{L}}{\partial A^{ij}{}_{\mu,\nu}} - 2a(\sqrt{-g}e^{[\mu}{}_i e^{\nu]}{}_j - e^{(0)[\mu}{}_i e^{(0)\nu]}{}_j)\right] \ . \tag{5·14}$$

More explicitly, the superpotential $\boldsymbol{\Psi}_{\mu}{}^{\nu\rho}$ is expressed as

$$\begin{aligned} \boldsymbol{\Psi}_{\mu}{}^{\nu\rho} &= -(2a/\sqrt{-g})g_{\mu\lambda}\partial_\sigma(gg^{\lambda[\nu}g^{\rho]\sigma}) \\ &\quad +2a\sqrt{-g}[\delta_{\mu}{}^{[\nu}(e^{\rho]}{}_i e^{\lambda i}{}_{,\lambda} - e^{\lambda}{}_i e^{\rho]i}{}_{,\lambda}) + e^{[\nu}{}_i e^{\rho]i}{}_{,\mu}] \\ &\quad -2\boldsymbol{J}^{[ij][\nu\rho]}\Delta_{ij\mu} \ , \end{aligned} \tag{5·15}$$

where we have defined

$$\boldsymbol{J}^{ij\nu\rho} \stackrel{\rm def}{=} -\frac{1}{2}\frac{\partial \boldsymbol{L}_R}{\partial A_{ij\nu,\rho}} \ , \tag{5·16}$$

$$\Delta_{ij\mu} \stackrel{\rm def}{=} \frac{1}{2}e^k{}_\mu(C_{ijk} - C_{jik} - C_{kij}) \tag{5·17}$$

with

$$C_{ijk} \stackrel{\rm def}{=} e^\nu{}_j e^\lambda{}_k(\partial_\nu e_{i\lambda} - \partial_\lambda e_{i\nu}) \ . \tag{5·18}$$

For $\boldsymbol{\Sigma}_{ij}{}^{\mu\nu}$, we have

$$\boldsymbol{\Sigma}_{ij}{}^{\mu\nu} = 4a\left[\sqrt{-g}e^{[\mu}{}_i e^{\nu]}{}_j - e^{(0)[\mu}{}_i e^{(0)\nu]}{}_j\right] + 4\boldsymbol{J}_{[ij]}{}^{[\mu\nu]} \ . \tag{5·19}$$

The expression of $\boldsymbol{\Psi}_0{}^{0\alpha}$ and of $\boldsymbol{\Psi}_\alpha{}^{0\beta}$ in terms of $A, B, \cdots$ and F is given by

$$\begin{aligned} \boldsymbol{\Psi}_0{}^{0\alpha} &= 4a\frac{1}{r}\left(1 - h - \frac{AD}{\Delta}\right)\frac{x^\alpha}{r} + 3c_2\frac{hh'}{r^2}\frac{x^\alpha}{r} \ , \\ \boldsymbol{\Psi}_\alpha{}^{0\beta} &= 2a\left[A'B - C'(D+E)\right]\left(\delta^\beta_\alpha - \frac{x^\alpha x^\beta}{r^2}\right) - 2a\frac{1}{r}\frac{BD}{\Delta}\left(\delta^\beta_\alpha + \frac{x^\alpha x^\beta}{r^2}\right) \\ &\quad +2a\frac{1}{r}\frac{BF}{\Delta}\epsilon_{\alpha\beta\gamma}\frac{x^\gamma}{r} + 3c_2\frac{h'}{r^2}\left[\frac{BD}{\Delta}\left(\delta^\beta_\alpha - \frac{x^\alpha x^\beta}{r^2}\right) + \frac{BF}{\Delta}\epsilon_{\alpha\beta\gamma}\frac{x^\gamma}{r}\right] \\ &\quad -6c_2\frac{h}{r^2}\left[A'B - C'(D+E)\right]\frac{x^\alpha x^\beta}{r^2} \ , \end{aligned} \tag{5·20}$$

where we have defined $h' \stackrel{\rm def}{=} dh/dr, A' \stackrel{\rm def}{=} dA/dr$, etc. Also, we have

$$\begin{aligned}
\mathbf{\Sigma}_{(0)(a)}{}^{0\alpha} &= 2a\left[\frac{F^2-DE}{\Delta}\left(\delta^{\alpha}_{(a)}-\frac{x^{\alpha}x^{(a)}}{r^2}\right)-\frac{(D+E)F}{\Delta}\epsilon_{(a)\alpha\beta}\frac{x^{\beta}}{r}\right] \\
&\quad -6c_2\Big[\frac{1}{2}\frac{h'}{r}\frac{D(D+E)}{\Delta}\left(\delta^{\alpha}_{(a)}-\frac{x^{\alpha}x^{(a)}}{r^2}\right) \\
&\qquad +\frac{1}{2}\frac{h'}{r}\frac{(D+E)F}{\Delta}\epsilon_{(a)\alpha\beta}\frac{x^{\beta}}{r}+\frac{1}{\Delta}\frac{h}{r^2}\frac{x^{\alpha}x^{(a)}}{r^2}\Big]\ , \\
\mathbf{\Sigma}_{(a)(b)}{}^{0\alpha} &= \left(2a+3c_2\frac{h'}{r}\right)\Big[\frac{BD}{\Delta}\left(\delta^{\alpha}_{(a)}\frac{x^{(b)}}{r}-\delta^{\alpha}_{(b)}\frac{x^{(a)}}{r}\right) \\
&\qquad +\frac{BF}{\Delta}\left(\epsilon_{(a)\alpha\beta}\frac{x^{\beta}x^{(b)}}{r^2}-\epsilon_{(b)\alpha\beta}\frac{x^{\beta}x^{(a)}}{r^2}\right)\Big]\ . \qquad (5{\cdot}21)
\end{aligned}$$

5.1 Energy-Momentum

Equation (5·3) can be written as

$$M_{\mu} = \int \mathbf{\Psi}_{\mu}{}^{0\alpha} r^2 n_{\alpha} d\Omega\ , \qquad (5{\cdot}22)$$

with the aid of Eq. (5·12), where $d\Omega$ stands for the differential solid angle. We can show that

$$\begin{aligned}
M_0 &= 16\pi a \lim_{r\to\infty} r\left(1-h-\frac{AD}{\Delta}\right)\ , \\
M_{\alpha} &= 0\ , \qquad (5{\cdot}23)
\end{aligned}$$

by the use of Eq. (5·20). We look for the condition imposed on the functions A, B, C, D, E and F by the requirement

$$M_{\mu} = -\delta_{\mu}{}^{0} M c^2\ , \qquad (5{\cdot}24)$$

which implies the equality of the active gravitational mass and the inertial mass. The condition (5·24) is equivalent to

$$\lim_{r\to\infty} r\left(1-\frac{AD}{\Delta}\right) = \frac{r_0}{2}\ . \qquad (5{\cdot}25)$$

If we express A/A_{∞} and D/D_{∞} as

$$\frac{A}{A_{\infty}} = 1-\frac{h}{2}+P\ , \quad \frac{D}{D_{\infty}} = 1+Q\ , \qquad (5{\cdot}26)$$

then the relation $\lim_{r\to\infty} P = 0 = \lim_{r\to\infty} Q$ follows. Equation (5·25) is equivalent to the condition

$$\lim_{r\to\infty} r(P + Q + PQ) = 0 \ . \tag{5·27}$$

Equation (5·24) agrees with Eq. (5·6) of Ref. [3], which has been obtained for generic systems being at rest as a whole.

5.2 Angular Momentum

By virtue of Eqs. (5·12) and (5·13), Eqs. (5·4) and (5·5) can be rewritten as

$$\begin{aligned} L^{\mu\nu} &= \eta^{\nu\lambda} \int x^{\mu} \mathbf{\Psi}_{\lambda}{}^{0\alpha} r^2 n_{\alpha} d\Omega - (\mu \rightleftarrows \nu) \ , & (5·28) \\ S_{ij} &= \int \mathbf{\Sigma}_{ij}{}^{0\alpha} r^2 n_{\alpha} d\Omega \ . & (5·29) \end{aligned}$$

These lead to, upon using Eqs. (5·20) and (5·21),

$$L^{\mu\nu} = 0 \ , \quad S_{ij} = 0 \ , \tag{5·30}$$

which hold for any A, B, C, D, E and F satisfying the conditions (4·3) and (4·5).

6 Generators in $\overline{\text{Poincaré}}$ gauge theory

In this section, on the basis of the discussion in Refs. [4] and [5], we examine generators of internal Poincaré transformations and of general affine coordinate transformations for the Schwarzschild space-time in $\bar{\text{P}}$GT.

6.1 The case when $\{\psi^k, A^k{}_{\mu}, A^{kl}{}_{\mu}, \varphi\}$ is employed as the set of independent field variables

Let us denote the Lagrangians $\boldsymbol{L}$ and $\boldsymbol{L}_G$ expressed as functions of ψ^k, $A^k{}_{\mu}$, $A^{kl}{}_{\mu}$, φ and of their derivatives by $\hat{\boldsymbol{L}}$ and $\hat{\boldsymbol{L}}_G$, respectively. For this case, the generator $\hat{M}_k$ of *internal* translations and the generator $\hat{S}_{kl}$ of *internal* Lorentz transformations are [4] the dynamical energy-momentum and the total (=spin+orbital) angular momentum, respectively, and

they are expressed as

$$\hat{M}_k \stackrel{\rm def}{=} \int_\sigma {}^{\rm tot}\hat{\boldsymbol{T}}_k{}^\mu d\sigma_\mu\ , \tag{6·1}$$

$$\hat{S}_{kl} \stackrel{\rm def}{=} \int_\sigma {}^{\rm tot}\hat{\boldsymbol{S}}_{kl}{}^\mu d\sigma_\mu \tag{6·2}$$

with

$$\begin{aligned}
{}^{\rm tot}\hat{\boldsymbol{T}}_k{}^\mu &\stackrel{\rm def}{=} \frac{\partial\hat{\boldsymbol{L}}}{\partial\psi^k{}_{,\mu}} + i\frac{\partial\hat{\boldsymbol{L}}}{\partial\varphi_{,\mu}}P_k\varphi + \frac{\partial\hat{\boldsymbol{L}}}{\partial A^l{}_{\nu,\mu}}A_k{}^l{}_\nu\ , & (6·3)\\
{}^{\rm tot}\hat{\boldsymbol{S}}_{kl}{}^\mu &\stackrel{\rm def}{=} -2\left(\frac{\partial\hat{\boldsymbol{L}}}{\partial\psi^{[k}{}_{,\mu}}\psi_{l]} + \hat{\boldsymbol{F}}_{[k}{}^{\nu\mu}A_{l]\nu} + 2\hat{\boldsymbol{F}}_{[km}{}^{\nu\mu}A_{l]}{}^m{}_\nu + \frac{i}{2}\frac{\partial\hat{\boldsymbol{L}}}{\partial\varphi_{,\mu}}M_{kl}\varphi\right)\ , & (6·4)\\
\hat{\boldsymbol{F}}_k{}^{\mu\nu} &\stackrel{\rm def}{=} \frac{\partial\hat{\boldsymbol{L}}}{\partial A^k{}_{\mu,\nu}}\ ,\quad \hat{\boldsymbol{F}}_{kl}{}^{\mu\nu} \stackrel{\rm def}{=} \frac{\partial\hat{\boldsymbol{L}}}{\partial A^{kl}{}_{\mu,\nu}}\ , & (6·5)\\
\hat{\boldsymbol{\Sigma}}_{kl}{}^{\mu\nu} &\stackrel{\rm def}{=} -2\left[\hat{\boldsymbol{F}}_{kl}{}^{\mu\nu} - 2a\left(\sqrt{-g}e^{[\mu}{}_k e^{\nu]}{}_l - e^{(0)[\mu}{}_k e^{(0)\nu]}{}_l\right)\right]\ . & (6·6)
\end{aligned}$$

Also, there are the identities

$$-\frac{\delta\hat{\boldsymbol{L}}}{\delta A^k{}_\mu} + {}^{\rm tot}\hat{\boldsymbol{T}}_k{}^\mu \equiv \partial_\nu\hat{\boldsymbol{F}}_k{}^{\mu\nu}\ , \tag{6·7}$$

$$\frac{\delta\hat{\boldsymbol{L}}}{\delta A^{kl}{}_\mu} + \frac{1}{2}{}^{\rm tot}\hat{\boldsymbol{S}}_{kl}{}^\mu \equiv \frac{1}{2}\partial_\nu\hat{\boldsymbol{\Sigma}}_{kl}{}^{\mu\nu}\ . \tag{6·8}$$

The functions $\hat{\boldsymbol{F}}_k{}^{\mu\nu}$ and $\hat{\boldsymbol{F}}_{kl}{}^{\mu\nu}$ have the expressions

$$\begin{aligned}
\hat{\boldsymbol{F}}_k{}^{\mu\nu} &= (2a/\sqrt{-g})e_{k\rho}\partial_\sigma\left[(-g)g^{[\mu\rho}g^{\nu]\sigma}\right]\\
&\quad +2a\sqrt{-g}\left[e^{[\mu}{}_k\left(e^{\nu]l}e^\lambda{}_{l,\lambda} - e^{\lambda l}e^{\nu]}{}_{l,\lambda}\right) + e^\lambda{}_k e^{[\mu l}e^{\nu]}{}_{l,\lambda}\right]\ ,\\
\hat{\boldsymbol{F}}_{kl}{}^{\mu\nu} &= -2\hat{\boldsymbol{J}}_{[kl]}{}^{[\mu\nu]} + \hat{\boldsymbol{F}}_{[k}{}^{\mu\nu}\psi_{l]}
\end{aligned} \tag{6·9}$$

with

$$\hat{\boldsymbol{J}}_{[kl]}{}^{[\mu\nu]} \stackrel{\rm def}{=} -\frac{1}{2}\frac{\partial\hat{\boldsymbol{L}}_R}{\partial A^{kl}{}_{\mu,\nu}}\ . \tag{6·10}$$

The expressions of $\hat{\boldsymbol{F}}_{(0)}{}^{0\alpha}$ and of $\hat{\boldsymbol{F}}_{(a)}{}^{0\alpha}$ in terms of $A, B, \cdots$ and F are given by

$$\begin{aligned}
\hat{\boldsymbol{F}}_{(0)}{}^{0\alpha} &= 4a\frac{1}{r}\left(A - \frac{D}{\Delta}\right)\frac{x^\alpha}{r}\ ,\\
\hat{\boldsymbol{F}}_{(a)}{}^{0\alpha} &= -4a\frac{1}{r}C\frac{x^{(a)}x^\alpha}{r^2} + 2a\left\{D\left[A'B - C'(D+E)\right] - \frac{1}{r}\frac{B}{\Delta}\right\}\left(\delta^\alpha_{(a)} - \frac{x^{(a)}x^\alpha}{r^2}\right)\\
&\quad +2aF\left[A'B - C'(D+E)\right]\epsilon_{(a)\alpha\beta}\frac{x^\beta}{r}\ .
\end{aligned} \tag{6·11}$$

For $\hat{\boldsymbol{\Sigma}}_{kl}{}^{\mu\nu}$, we have

$$
\begin{aligned}
\hat{\boldsymbol{\Sigma}}_{(0)(a)}{}^{0\alpha} &= 2a\left[\frac{1}{\Delta}(F^2-DE)\left(\delta^\alpha_{(a)}-\frac{x^\alpha x^{(a)}}{r^2}\right)-\frac{(D+E)F}{\Delta}\epsilon_{(a)\alpha\beta}\frac{x^\beta}{r}\right] \\
&\quad -3c_2\Bigg[\frac{h'}{r}\frac{D(D+E)}{\Delta}\left(\delta^\alpha_{(a)}-\frac{x^\alpha x^{(a)}}{r^2}\right)+\frac{h'}{r}\frac{(D+E)F}{\Delta}\epsilon_{(a)\alpha\beta}\frac{x^\beta}{r} \\
&\quad +\frac{2}{\Delta}\frac{h}{r^2}\frac{x^\alpha x^{(a)}}{r^2}\Bigg]-4a\frac{1}{r}\left(A-\frac{D}{\Delta}\right)\frac{x^\alpha}{r}\psi_{(a)} \\
&\quad +2a\Bigg(-\frac{2}{r}C\frac{x^\alpha x^{(a)}}{r^2}+\left[A'B-C'(D+E)\right]F\epsilon_{(a)\alpha\beta}\frac{x^\beta}{r} \\
&\quad +\left\{\left[A'B-C'(D+E)\right]D-\frac{1}{r}\frac{B}{\Delta}\right\}\left(\delta^\alpha_{(a)}-\frac{x^\alpha x^{(a)}}{r^2}\right)\Bigg)\psi_{(0)}\ , \\
\hat{\boldsymbol{\Sigma}}_{(a)(b)}{}^{0\alpha} &= \left(2a+3c_2\frac{h'}{r}\right)\Bigg[\frac{BD}{\Delta}\left(\delta^\alpha_{(a)}\frac{x^{(b)}}{r}-\delta^\alpha_{(b)}\frac{x^{(a)}}{r}\right) \\
&\quad +\frac{BF}{\Delta}\left(\epsilon_{(a)\alpha\beta}\frac{x^\beta x^{(b)}}{r^2}-\epsilon_{(b)\alpha\beta}\frac{x^\beta x^{(a)}}{r^2}\right)\Bigg] \\
&\quad -\Bigg[2a\Bigg(-\frac{2}{r}C\frac{x^\alpha x^{(a)}}{r^2}+\left[A'B-C'(D+E)\right]F\epsilon_{(a)\alpha\beta}\frac{x^\beta}{r} \\
&\quad +\left\{\left[A'B-C'(D+E)\right]D-\frac{1}{r}\frac{B}{\Delta}\right\}\left(\delta^\alpha_{(a)}-\frac{x^\alpha x^{(a)}}{r^2}\right)\Bigg)\psi_{(b)} \\
&\quad -((a)\rightleftarrows(b))\Bigg]\ . \qquad (6\cdot12)
\end{aligned}
$$

6.1.1 Energy-Momentum

By using Eq. (6·11) and the expression

$$
\hat{M}_k=\int\hat{\boldsymbol{F}}_k{}^{0\alpha}r^2n_\alpha d\Omega \qquad (6\cdot13)
$$

which follows from Eqs. (6·1) and (6·7), we can obtain

$$
\begin{aligned}
\hat{M}_{(0)} &= 16\pi a\lim_{r\to\infty}r\left(A-\frac{D}{\Delta}\right)\ , \\
\hat{M}_{(a)} &= 0\ . \qquad (6\cdot14)
\end{aligned}
$$

We have the relation [4] $\hat{M}_k=e^{(0)\mu}{}_kM_\mu$ for generic space-times satisfying suitable asymptotic condition. Here, M_μ is the energy-momentum vector of the total system, which

agrees[10] with the canonical energy-momentum in PGT given by Eq. (5·22). In view of this, we require the relation

$$\hat{M}_k = -e^{(0)\mu}{}_k M c^2 \delta_\mu{}^0 \ , \tag{6·15}$$

following the requirement (5·24) in PGT. This is equivalent to

$$\lim_{r\to\infty} r\left(\frac{D}{D_\infty} - \frac{A}{A_\infty}\right) = \frac{r_0}{2} \ . \tag{6·16}$$

If we express A/A_∞ and D/D_∞ as

$$\frac{A}{A_\infty} = 1 - \frac{h}{2} + U \ , \quad \frac{D}{D_\infty} = 1 + V \ , \tag{6·17}$$

then the relation $\lim_{r\to\infty} U = 0 = \lim_{r\to\infty} V$ follows. Equation (6·16) is equivalent to the condition

$$\lim_{r\to\infty} r(U - V) = 0 \ . \tag{6·18}$$

6.1.2 Angular Momentum

As is known from Ref. [4], the angular momentum in $\bar{\text{P}}$GT depends on the asymptotic behavior of the field ψ,[11] and the following

$$\begin{aligned}
\psi^k &= e^{(0)k}{}_\mu x^\mu + \psi^{(0)k} + O(1/r^\beta) \ , && (6·19)\\
\psi^k{}_{,\mu} &= e^{(0)k}{}_\mu + O(1/r^{1+\beta}) \ , && (6·20)
\end{aligned}$$

is quite natural, which we shall assume in this paper. Here, $\psi^{(0)k}$ and β are a constant and a positive constant, respectively, and $O(1/r^n)$ with positive n denotes a term for which $r^n O(1/r^n)$ remains finite for $r \to \infty$; a term $O(1/r^n)$ may of course also be zero.

The angular momentum $\hat{S}_{kl}$ is expressed as

$$\hat{S}_{kl} = \int \hat{\mathbf{\Sigma}}_{kl}{}^{0\alpha} r^2 n_\alpha d\Omega \ , \tag{6·21}$$

which follows from Eqs. (6·2) and (6·8). From Eqs. (6·12), (6·15) and (6·21), we obtain the following:

$$\begin{aligned}
\hat{S}_{(0)(a)} &= -\hat{M}_{(0)}\psi^{(0)}{}_{(a)} = 2\psi^{(0)}{}_{[(0)}\hat{M}_{(a)]} \ , && (6·22)\\
\hat{S}_{(a)(b)} &= 0 = 2\psi^{(0)}{}_{[(a)}\hat{M}_{(b)]} \ , && (6·23)
\end{aligned}$$

[10] In view of this agreement, we use the same symbol M_μ for these two energy-momenta.

[11] We choose this field to be regular everywhere in the finite region of the space-time. This is possible, because the field ψ can be chosen arbitrarily as has been shown in Ref. [2].

when the conditions (6·18) and

$$U = O(1/r^2) \tag{6·24}$$

are both satisfied. Equations (6·22) and (6·23) show that $\hat{S}_{kl}$ agrees with the orbital angular momentum [4] around the origin of the internal space $\mathbf{R}^4$. These equations agree with Eq. (5·8) of Ref. [4], because the first term in the r.h.s. of this equation vanishes for the Schwarzschild metric.

6.1.3 Canonical Energy-Momentum and "Extended Orbital Angular Momentum"

The generator $\hat{M}^c{}_\mu$ of coordinate translations and the generator $\hat{L}_\mu{}^\nu$ of $GL(4,\mathbf{R})$ coordinate transformations are the canonical energy-momentum and[12] the "extended orbital angular momentum", respectively. They have the expressions

$$\hat{M}^c{}_\mu \stackrel{\text{def}}{=} \int_\sigma {}^{\text{tot}}\hat{\tilde{\boldsymbol{T}}}{}_\mu{}^\nu d\sigma_\nu \;, \tag{6·25}$$

$$\hat{L}_\mu{}^\nu \stackrel{\text{def}}{=} \int_\sigma \hat{\boldsymbol{M}}_\mu{}^{\nu\lambda} d\sigma_\lambda \;, \tag{6·26}$$

where we have defined[13]

$${}^{\text{tot}}\hat{\tilde{\boldsymbol{T}}}{}_\mu{}^\nu \stackrel{\text{def}}{=} \hat{\boldsymbol{L}}\delta_\mu{}^\nu - \left(\hat{\boldsymbol{F}}_k{}^{\lambda\nu} A^k{}_{\lambda,\mu} + \hat{\boldsymbol{F}}_{kl}{}^{\lambda\nu} A^{kl}{}_{\lambda,\mu} + \frac{\partial \hat{\boldsymbol{L}}}{\partial \psi^k{}_{,\nu}} \psi^k{}_{,\mu} + \frac{\partial \hat{\boldsymbol{L}}}{\partial \varphi_{,\nu}} \varphi_{,\mu} \right) , \tag{6·27}$$

$$\hat{\boldsymbol{M}}_\mu{}^{\nu\lambda} \stackrel{\text{def}}{=} -2 \left(x^\nu \; {}^{\text{tot}}\hat{\tilde{\boldsymbol{T}}}{}_\mu{}^\lambda - \hat{\boldsymbol{\Psi}}_\mu{}^{\nu\lambda} \right) \tag{6·28}$$

with

$$\hat{\boldsymbol{\Psi}}_\mu{}^{\nu\lambda} \stackrel{\text{def}}{=} \hat{\boldsymbol{F}}_k{}^{\nu\lambda} A^k{}_\mu + \hat{\boldsymbol{F}}_{kl}{}^{\nu\lambda} A^{kl}{}_\mu \;. \tag{6·29}$$

From Eq. (6·25) and the identity

$$-\frac{\delta \hat{\boldsymbol{L}}}{\delta A^k{}_\nu} A^k{}_\mu - \frac{\delta \hat{\boldsymbol{L}}}{\delta A^{kl}{}_\nu} A^{kl}{}_\mu + {}^{\text{tot}}\hat{\tilde{\boldsymbol{T}}}{}_\mu{}^\nu \equiv \partial_\lambda \hat{\boldsymbol{\Psi}}_\mu{}^{\nu\lambda} \;, \tag{6·30}$$

the expression

$$\hat{M}^c{}_\mu = \int \hat{\boldsymbol{\Psi}}_\mu{}^{0\alpha} r^2 n_\alpha d\Omega \;, \tag{6·31}$$

[12] Note that the anti-symmetric part $\hat{L}_{[\mu\nu]} \stackrel{\text{def}}{=} \hat{L}_{[\mu}{}^\lambda \eta_{\lambda\nu]}$ is the orbital angular momentum.

[13] The energy-momentum density of the gravitational field is defined by the r.h.s. of Eq. (6·27) with $\hat{\boldsymbol{L}}$ being replaced with $\hat{\boldsymbol{L}}_G$. This density does not give vanishing energy-momentum when it is integrated over a space-like surface.

is obtained. By evaluating the r.h.s. of this, we obtain

$$\hat{M}^c{}_\mu = 0 \ , \tag{6·32}$$

when the condition

$$\lim_{r\to\infty} rU = 0 \ . \tag{6·33}$$

is satisfied. For $\hat{L}_\mu{}^\nu$, we have

$$\hat{L}_\mu{}^\nu = -2\int x^\nu \hat{\boldsymbol{\Psi}}_\mu{}^{0\alpha} n_\alpha d\Omega \ , \tag{6·34}$$

which follows from Eqs. (6·26) and (6·30). By evaluating the r.h.s. of this, we obtain

$$\hat{L}_0{}^0 = 0 \ , \quad \hat{L}_\mu{}^\nu = 0 \ , \quad \mu \neq \nu \ , \tag{6·35}$$

when the conditions (6·18) and (6·33) are both satisfied. Also, we have

$$\hat{L}_1{}^1 = \hat{L}_2{}^2 = \hat{L}_3{}^3 = 0 \ , \tag{6·36}$$

if the conditions (6·33) and

$$U' = O(1/r^{1+\gamma}) \tag{6·37}$$

with γ being a positive constant are both satisfied. It is also known that the orbital angular momentum $\hat{L}_{[\mu\nu]}$ vanishes if the condition (6·24) is satisfied.

6.2 The case when $\{\psi^k, e^k{}_\mu, A^{kl}{}_\mu, \varphi\}$ is employed as the set of independent field variables

Let us denote the Lagrangians $\boldsymbol{L}$ and $\boldsymbol{L}_G$ expressed as functions of $\psi^k, e^k{}_\mu, A^{kl}{}_\mu, \varphi$ and of their derivatives by $\check{\boldsymbol{L}}$ and $\check{\boldsymbol{L}}_G$, respectively.

For the dynamical energy-momentum $\check{M}_k$, we have [5]

$$\check{M}_k \stackrel{\text{def}}{=} \int_\sigma {}^{\text{tot}}\check{\boldsymbol{T}}_k{}^\mu d\sigma_\mu \equiv 0 \tag{6·38}$$

with

$${}^{\text{tot}}\check{\boldsymbol{T}}_k{}^\mu \stackrel{\text{def}}{=} \frac{\partial \check{\boldsymbol{L}}}{\partial \psi^k{}_{,\mu}} + i\frac{\partial \check{\boldsymbol{L}}}{\partial \varphi_{,\mu}} P_k \varphi \equiv 0 \ . \tag{6·39}$$

The generator $\check{S}_{kl}$ of internal Lorentz transformations is expressed as

$$\check{S}_{kl} \stackrel{\text{def}}{=} \int_\sigma {}^{\text{tot}}\check{\boldsymbol{S}}_{kl}{}^{\mu} d\sigma_\mu \tag{6·40}$$

with

$${}^{\text{tot}}\check{\boldsymbol{S}}_{kl}{}^{\mu} \stackrel{\text{def}}{=} -2\frac{\partial\check{\boldsymbol{L}}}{\partial\psi^{[k}{}_{,\mu}}\psi_{l]} - 2\frac{\partial\check{\boldsymbol{L}}}{\partial e^{[k}{}_{\nu,\mu}}e_{l]\nu} - 4\frac{\partial\check{\boldsymbol{L}}}{\partial A^{[k}{}_{m\nu,\mu}}A_{l]m\nu} - i\frac{\partial\check{\boldsymbol{L}}}{\partial\varphi_{,\mu}}M_{kl}\varphi \ . \tag{6·41}$$

Also, we have the identity

$$\frac{\delta\check{\boldsymbol{L}}}{\delta A^{kl}{}_{\mu}} + \frac{1}{2}{}^{\text{tot}}\check{\boldsymbol{S}}_{kl}{}^{\mu} \equiv \frac{1}{2}\partial_\nu\check{\boldsymbol{\Sigma}}_{kl}{}^{\mu\nu} \tag{6·42}$$

with

$$\check{\boldsymbol{\Sigma}}_{kl}{}^{\mu\nu} \stackrel{\text{def}}{=} -2\frac{\partial\check{\boldsymbol{L}}}{\partial A^{kl}{}_{\mu,\nu}} + 4a\left(\sqrt{-g}e^{[\mu}{}_{k}e^{\nu]}{}_{l} - e^{(0)[\mu}{}_{k}e^{(0)\nu]}{}_{l}\right) \ . \tag{6·43}$$

Equation (6·40) can be rewritten as

$$\check{S}_{kl} = \int \check{\boldsymbol{\Sigma}}_{kl}{}^{0\alpha} r^2 n_\alpha d\Omega \ , \tag{6·44}$$

with the aid of Eq. (6·42). The integrand in Eq. (6·44) agrees with that in Eq. (5·29), and it follows that

$$\check{S}_{kl} = S_{kl} = 0 \ , \tag{6·45}$$

which holds for any A, B, C, D, E and F satisfying the conditions (4·3) and (4·5).

The canonical energy-momentum $\check{M}^c{}_{\mu}$ is expressed as

$$\check{M}^c{}_{\mu} \stackrel{\text{def}}{=} \int_\sigma {}^{\text{tot}}\check{\tilde{\boldsymbol{T}}}_{\mu}{}^{\nu} d\sigma_\nu \tag{6·46}$$

with[14]

$${}^{\text{tot}}\check{\tilde{\boldsymbol{T}}}_{\mu}{}^{\nu} \stackrel{\text{def}}{=} \check{\boldsymbol{L}}\delta_\mu{}^\nu - \left(\frac{\partial\check{\boldsymbol{L}}}{\partial\psi^k{}_{,\nu}}\psi^k{}_{,\mu} + \frac{\partial\check{\boldsymbol{L}}}{\partial e^k{}_{\lambda,\nu}}e^k{}_{\lambda,\mu} + \frac{\partial\check{\boldsymbol{L}}}{\partial A^{kl}{}_{\lambda,\nu}}A^{kl}{}_{\lambda,\mu} + \frac{\partial\check{\boldsymbol{L}}}{\partial\varphi_{,\nu}}\varphi_{,\mu}\right) \ . \tag{6·47}$$

There is the identity

$$-\frac{\delta\check{\boldsymbol{L}}}{\delta e^k{}_{\nu}}e^k{}_{\mu} - \frac{\delta\check{\boldsymbol{L}}}{\delta A^{kl}{}_{\nu}}A^{kl}{}_{\mu} + {}^{\text{tot}}\check{\tilde{\boldsymbol{T}}}_{\mu}{}^{\nu} \equiv \partial_\lambda\check{\boldsymbol{\Psi}}_{\mu}{}^{\nu\lambda} \ , \tag{6·48}$$

[14]The energy-momentum density of the gravitational field is defined by the r.h.s. of Eq. (6·47) with $\check{\boldsymbol{L}}$ being replaced with $\check{\boldsymbol{L}}_G$. This density agrees with that of PGT, if the "intrinsic" energy-momentum P_k of the field φ is vanishing.

where we have defined

$$\check{\boldsymbol{\Psi}}_{\mu}{}^{\nu\lambda} \stackrel{\mathrm{def}}{=} \frac{\partial \check{\boldsymbol{L}}}{\partial e^{k}{}_{\nu,\lambda}} e^{k}{}_{\mu} + \frac{\partial \check{\boldsymbol{L}}}{\partial A^{kl}{}_{\nu,\lambda}} A^{kl}{}_{\mu} \ . \tag{6·49}$$

From Eqs. (5·3), (5·7), (5·12), (6·46), (6·48) and (6·49), we can show that

$$\check{M}^{c}{}_{\mu} = M_{\mu} \ . \tag{6·50}$$

When the requirement (5·24) is imposed, the functions A, B, C, D, E and F agree with as those in the case of PGT.

The "extended orbital angular momentum", which is the generator of $GL(4, \mathbf{R})$ coordinate transformations, has the expression

$$\check{L}_{\mu}{}^{\nu} \stackrel{\mathrm{def}}{=} \int_{\sigma} \check{\boldsymbol{M}}_{\mu}{}^{\nu\lambda} d\sigma_{\lambda} \tag{6·51}$$

with

$$\check{\boldsymbol{M}}_{\mu}{}^{\nu\lambda} \stackrel{\mathrm{def}}{=} -2 \left(x^{\nu} \ {}^{\mathrm{tot}}\check{\tilde{\boldsymbol{T}}}_{\mu}{}^{\lambda} - \check{\boldsymbol{\Psi}}_{\mu}{}^{\nu\lambda} \right) \ . \tag{6·52}$$

The integral in Eq. (6·51) is evaluated to give

$$\begin{aligned} \check{L}_{[\mu\nu]} &\stackrel{\mathrm{def}}{=} \check{L}_{[\mu}{}^{\lambda}\eta_{\lambda\nu]} = 0 \ , \\ \check{L}_{\mu}{}^{0} &= -2x^{0} M_{\mu} \ , \quad \check{L}_{0}{}^{\alpha} = 0 \ , \\ \check{L}_{\alpha}{}^{\beta} &= \frac{16\pi}{3} \delta_{\alpha}{}^{\beta} \lim_{r\to\infty} \left\{ \frac{2ar^{2}BD}{\Delta} + 3c_{2}r_{0}[A'B - C'(D+E)] \right\} \ , \end{aligned} \tag{6·53}$$

the first of which implies that the orbital angular momentum is vanishing. We have

$$\check{L}_{1}{}^{1} = \check{L}_{2}{}^{2} = \check{L}_{3}{}^{3} = 0 \ , \tag{6·54}$$

if

$$P = -\frac{1}{2^{3}} h^{2} - \frac{1}{2^{4}} h^{3} - \frac{5}{2^{7}} h^{4} + Z \tag{6·55}$$

with Z satisfying $Z = O(1/r^{6})$ and $\lim_{r\to\infty} r^{3} Z' = 0$.

7 Summary and comments

The results obtained in §§3∼6 can be summarized as follows:

1. Both in PGT and in $\bar{\rm P}$GT, the Schwarzschild space-time expressed in terms of the Schwarzschild coordinates is obtainable as a torsionless exact solution of gravitational field equations with a spinless point-like source located at the origin, if and only if the condition (3·3) is satisfied. This and the fact mentioned in the footnote on page 7 show that the Lagrangian L_R with the condition (3·3) is favorable in various respects.

2. Spherically symmetric vierbeins (4·2) have been considered by choosing the gauge of internal Lorentz group in a way such that the function A is independent of x^0.

3. For PGT, the equality of the active gravitational mass and the inertial mass is satisfied, if and only if the condition (5·25), which is equivalent to Eq. (5·27), is satisfied. Also, the spin angular momentum S_{kl} and orbital angular momentum $L^{\mu\nu}$ both vanish for any A, B, C, D, E and F satisfying the conditions (4·3) and (4·5).

4. For $\bar{\rm P}$GT, generators depend on the choice of the set of independent field variables.

   (A) The case when $\{\psi^k, A^k{}_\mu, A^{kl}{}_\mu, \varphi\}$ is employed as the set of independent field variables.
   For this case, dynamical energy-momentum $\hat{M}_k$, which is the generator of internal translations, has the expression (6·15),[15] if the condition (6·16) (or equivalently the condition (6·18)) is satisfied. For the total angular momentum $\hat{S}_{kl}$, we have $\hat{S}_{kl} = 2\psi^{(0)}{}_{[k}\hat{M}_{l]}$, if the conditions (6·18) and (6·24) are both satisfied. This shows that the total angular momentum of this space-time is only the orbital angular momentum around the origin of the internal space $\mathbf{R}^4$, which is quite reasonable because this space-time is a static spherically symmetric one with a static spinless point-like source. The canonical energy-momentum $\hat{M}^c{}_\mu$ vanishes, if the condition (6·33) is satisfied. The "extended orbital angular momentum" $\hat{L}_\mu{}^\nu$ vanishes, if the conditions (6·33) and (6·37) are both satisfied. Thus, reasonable energy-momentum and angular momentum are obtained as the generators of the *internal* Poincaré transformations and the generators of general affine *coordinate* transformations vanish, if the conditions (6·18), (6·24), (6·33) and (6·37) are all satisfied.

[15] Remind the paragraph just below Eq. (6·14) and note that the relation $M_\mu = -Mc^2\delta_\mu{}^0$ gives the equality of the active gravitational mass and the inertial mass.

(B) The case when $\{\psi^k, e^k{}_\mu, A^{kl}{}_\mu, \varphi\}$ is employed as the set of independent field variables.

For this case, the dynamical energy-momentum $\check{M}_k$ identically vanish, [5] and also the spin angular momentum $\check{S}_{kl}$ and the orbital angular momentum $\check{L}_{[\mu\nu]}$ both vanish for any A, B, C, D, E and F satisfying the conditions (4·3) and (4·5). The canonical energy-momentum $\check{M}^c{}_\mu$ agrees with the energy-momentum M_μ in PGT, and the active gravitational mass is equal to the inertial mass, if the condition (5·25) is satisfied. The "extended orbital angular momentum" $\check{L}_\mu{}^\nu$ vanishes, if the condition (6·55) is satisfied.

Finally, we would like to add several comments:

[1] In general relativity, the same expression as (3·2) has been obtained[10] for the energy-momentum density of the source of the Schwarzschild space-time. Thus, the gravitational field of this space-time can be interpreted to be produced by a point-like particle with mass M located at $\boldsymbol{x} = \boldsymbol{0}$. The regularization scheme employed is crucial in giving this result.[10] In the present paper, we have developed our discussion on the premise that this interpretation and the regularization scheme are both applicable also to PGT and to $\bar{\text{P}}$GT.

[2] For the Schwarzschild space-time, the Lorentz gauge potentials $A^{kl}{}_\mu$ agrees with the Ricci rotation coefficients $\Delta^{kl}{}_\mu$, and the results given in §5. and in §6. show that reasonable energy-momenta and angular momenta are obtainable, even if the condition [3, 4]

$$\Delta^{kl}{}_\mu = O\left(\frac{1}{r^{1+\alpha}}\right) \ , \ \ \Delta^{kl}{}_{\mu,(m)} = O\left(\frac{1}{r^{2+\alpha}}\right) \ , \ \ m = 1, 2, \ \ \alpha > 0 \ , \tag{7·1}$$

or the weaker condition [3, 4]

$$\Delta^{kl}{}_\mu = \frac{M^{kl}{}_\mu}{r} + L^{kl}{}_\mu \tag{7·2}$$

with $M^{kl}{}_\mu$ being a constant and

$$L^{kl}{}_\mu = O\left(\frac{1}{r^2}\right) \ , \ \ L^{kl}{}_{\mu,(m)} = O\left(\frac{1}{r^3}\right) \ , \ \ m = 1, 2 \ , \tag{7·3}$$

are not satisfied. Here, $\Delta^{kl}{}_{\mu,(m)}$, for example, denotes the m-th order partial derivative of $\Delta^{kl}{}_\mu$ with respect to x^λ.

[3] As we have seen in the summary, the requirement that the active gravitational mass is equal to the inertial mass restricts the behavior of the vierbeins at spatial infinity. The physical meaning of vierbeins violating this equality is not yet clear. In this connection, it is worth mentioning the following: (1) There is a similar situation also in new general relativity. [12, 13] (2) Also in the framework of general relativity, the equality of the active gravitational mass and the inertial gravitational mass is violated for the Schwarzschild metric expressed with the use of a certain coordinate system. [14]

[4] Discussions in §5. and in §6. present us with a test of definitions of energy-momenta and of angular momenta introduced in Ref. [3] and in Ref. [4], and the results summarized in 3. and in 4. support them.

A Irreducible components of T_{klm} and of R_{klmn}

Irreducible components of T_{klm} and of R_{klmn} are the following:

$$t_{klm} \stackrel{\text{def}}{=} \frac{1}{2}(T_{klm} + T_{lkm}) + \frac{1}{6}(\eta_{mk}v_l + \eta_{ml}v_k) - \frac{1}{3}\eta_{kl}v_m \ , \tag{A·1}$$

$$v_k \stackrel{\text{def}}{=} T^l{}_{lk} \ , \tag{A·2}$$

$$a_k \stackrel{\text{def}}{=} \frac{1}{6}\epsilon_{klmn}T^{lmn} \ , \tag{A·3}$$

$$A_{klmn} \stackrel{\text{def}}{=} \frac{1}{6}(R_{klmn} + R_{kmnl} + R_{knlm} + R_{lmkn} + R_{lnmk} + R_{mnkl}) \ , \tag{A·4}$$

$$B_{klmn} \stackrel{\text{def}}{=} \frac{1}{4}(W_{klmn} + W_{mnkl} - W_{knlm} - W_{lmkn}) \ , \tag{A·5}$$

$$C_{klmn} \stackrel{\text{def}}{=} \frac{1}{2}(W_{klmn} - W_{mnkl}) \ , \tag{A·6}$$

$$E_{kl} \stackrel{\text{def}}{=} \frac{1}{2}(R_{kl} - R_{lk}) \ , \tag{A·7}$$

$$I_{kl} \stackrel{\text{def}}{=} \frac{1}{2}(R_{kl} + R_{lk}) - \frac{1}{4}\eta_{kl}R \ , \tag{A·8}$$

$$R \stackrel{\text{def}}{=} \eta^{kl}R_{kl} \tag{A·9}$$

with

$$W_{klmn} \stackrel{\text{def}}{=} R_{klmn} - \frac{1}{2}(\eta_{km}R_{ln} + \eta_{ln}R_{km} - \eta_{kn}R_{lm} - \eta_{lm}R_{kn})$$

$$+\frac{1}{6}(\eta_{km}\eta_{ln} - \eta_{lm}\eta_{kn})R \ , \tag{A·10}$$

$$R_{kl} \stackrel{\text{def}}{=} \eta^{mn} R_{kmln} \ . \tag{A·11}$$

B Vierbeins giving the Schwarzschild metric

Vierbeins having the expression (4·2) are fixed by the functions A, B, C, D, E and F. We write down here an example of the set of $A, B, \cdots$ and F which gives the Schwarzschild metric:

$$A = \pm\left(1 - \frac{h}{2} + Jh^{\omega}\right) \ , \tag{B·1}$$

$$D = \pm\frac{1}{1 + K(x^0)h^{\omega}} \ , \tag{B·2}$$

$$C = \sqrt{A^2 - 1 + h} \ , \quad F = \sqrt{1 - D^2} \ , \tag{B·3}$$

$$B = \pm\frac{C}{1 - h} \ , \tag{B·4}$$

$$E = \pm\frac{A}{1 - h} - D \tag{B·5}$$

with the double signs in the expressions for B and E in same order.[16] Also, ω, J and K are a real constant, a non-negative real constant and a non-negative real valued function of x^0, respectively.

The function A in the above is independent of x^0, and we have the following:

$$\Delta = \pm 1 \ , \tag{B·6}$$

$$A_{\infty} = \pm 1 \ , \ \text{if and only if } J = 0 \ \text{ or } \omega > 0 \ , \tag{B·7}$$

$$D_{\infty} = \pm 1 \ , \ \text{if and only if } K \equiv 0 \ \text{ or } \omega > 0 \ , \tag{B·8}$$

where the double signs in Eq. (B·6), Eq. (B·7) and Eq. (B·8) are in same order as those in Eqs. (B·4) and (B·5), Eq. (B·1) and Eq. (B·2), respectively. The restrictions imposed on ω, J and K by the asymptotic conditions (5·27), (6·18), (6·24), (6·33), (6·37), etc. are easily known.

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[16] Otherwise, the order of the double signs in Eqs. (B·1)∼ (B·5) is arbitrary.

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